Algebraic identities associated with KP and AKNS hierarchies *

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Abstract

Explicit KP and AKNS hierarchy equations can be constructed from a certain set of algebraic identities involving a quasi-shuffle product.

1 Introduction

The equations of the KP hierarchy are well-known to possess multi-soliton solutions. According to Okhuma and Wadati [1], these solutions can be expressed as formal power series (in some indeterminate). Substitution into hierarchy equations leads to algebraic sum identities of a special kind. The structure of these identities has been explored in [2] and abstracted to a certain algebra which we briefly recall in section 2.[1] Moreover, a map Φ from the latter algebra to the algebra of pseudo-differential operators underlying the Gelfand-Dickey formulation [4] of the KP hierarchy was constructed, which maps a certain set of algebraic identities to KP hierarchy equations, and the whole KP hierarchy is actually obtained in this way. We briefly recall this map in section 3. In section 4 we show that, quite surprisingly, the same set of identities is also related to the AKNS hierarchy in a similar way. In particular, relations between AKNS and KP emerge from this relation, as will be demonstrated in section 5.

2 The algebra

Let $\mathcal{A} = \bigoplus_{r \geq 1} \mathcal{A}^r$ be a graded linear space over a field $\mathbb{K}$ of characteristic zero, supplied with two products $\prec : \mathcal{A}^r \times \mathcal{A}^s \to \mathcal{A}^{r+s}$ and $\bullet : \mathcal{A}^r \times \mathcal{A}^s \to \mathcal{A}^{r+s-1}$ which are associative and also

[1] These identities are in fact identities of quasi-symmetric functions, see [3] for example.

mixed associative. We assume[2] that $\mathcal{A}$ is generated by $\mathcal{A}^1$ via the product $\prec$. Now we define a *quasi-shuffle product* (see, e.g., [5]) $\circ$ in $\mathcal{A}$ by

$$\begin{aligned}
A \circ B &= A \prec B + B \prec A + A \bullet B && (1)\\
A \circ (B \prec \alpha) &= A \prec B \prec \alpha + B \prec (A \circ \alpha) + A \bullet B \prec \alpha && (2)\\
(A \prec \alpha) \circ B &= A \prec (\alpha \circ B) + B \prec A \prec \alpha + A \bullet B \prec \alpha && (3)\\
(A \prec \alpha) \circ (B \prec \beta) &= A \prec (\alpha \circ (B \prec \beta)) + B \prec ((A \prec \alpha) \circ \beta) \\
&\quad +(A \bullet B) \prec (\alpha \circ \beta) && (4)
\end{aligned}$$

for all $A, B \in \mathcal{A}^1$ and $\alpha, \beta \in \mathcal{A}$. This is another associative product in $\mathcal{A}$ which, however, is *not* mixed associative with the other products. If $(\mathcal{A}^1, \bullet)$ is commutative, then also $(\mathcal{A}, \circ)$ [2].

Let $\mathcal{A}(P)$ be the subalgebra generated by a single element $P \in \mathcal{A}^1$. We set $P^{\bullet n} := P \bullet \ldots \bullet P$ (n-fold product). Introducing the combined associative product $\succ := \prec + \bullet$, the next formula defines a further associative product in $\mathcal{A}(P)$,

$$\alpha \hat{\times} \beta := -\alpha \prec P \succ \beta \,. \tag{5}$$

In the following $\mathcal{I}$ denotes the set of identities in $\mathcal{A}(P)$ built from the elements $P^{\bullet n}$, $n = 1, 2, \ldots$ solely by use of the products $\circ$ and $\hat{\times}$.

3 From $\mathcal{A}(P)$ to the KP hierarchy

Let $\mathcal{R}$ denote the $\mathbb{K}$-algebra of formal pseudo-differential operators (ΨDOs) generated by

$$L = \partial + \sum_{n \geq 1} u_{n+1} \, \partial^{-n} \tag{6}$$

with coefficients from an associative algebra $\mathcal{B}$ (over $\mathbb{K}$), together with the projection $(\,)_{<0}$ to that part of a ΨDO containing only negative powers of the partial derivative operator ∂ (with respect to a variable x). Now

$$\ell(P) := L \,, \quad \ell(\alpha \prec P) := -\ell(\alpha)_{<0} \, L \,, \quad \ell(\alpha \bullet P) := \ell(\alpha) \, L \tag{7}$$

determines iteratively a map $\ell : \mathcal{A}(P) \to \mathcal{R}$. The map $\Phi : \mathcal{A}(P) \to \mathcal{B}$ defined by

$$\Phi(\alpha) := \mathrm{res}(\ell(\alpha)) \tag{8}$$

(where the residue of a ΨDO is the coefficient of its ∂^{-1} term) then has the following properties [2],

$$\Phi(\alpha \hat{\times} \beta) = \Phi(\alpha) \, \Phi(\beta) \,, \qquad \Phi(P^{\bullet n} \circ \alpha) = \delta_n \Phi(\alpha) \tag{9}$$

where δ_n are derivations with $\delta_n L := -[(L^n)_{<0}, L]$. They should be regarded as vector fields on the algebra $\mathcal{R}$. From [2] we recall:

Theorem. Writing $u_2 = \phi_x$ and imposing the flow equations $\delta_n = \partial_{t_n}$ (where $t_1 = x$), which imply $\Phi(P^{\bullet n}) = \phi_{t_n}$, all (combinations of) equations of the KP hierarchy lie in $\Phi(\mathcal{I})$. ■

[2]This assumption should have been added in [2].

We believe that *any* identity from $\mathcal{I}$ is mapped to a combination of KP hierarchy equations, so that the correspondence is actually one-to-one. This has not yet been proven, however. The set of algebraic identities specified in the theorem expresses the 'building rules' of *explicit* KP hierarchy equations, which are rather implicitly determined by the sequence of Lax equations $L_{t_n} = -[(L^n)_{<0}, L]$ [4]. For example, the algebraic identity

$$4\, P^{\bullet 3} \circ P - P^{\circ 4} - 6\, P \circ (P \hat{\times} P) = 6\, [P^{\bullet 2}, P]_{\hat{\times}} + 3\, P^{\bullet 2} \circ P^{\bullet 2} \tag{10}$$

is mapped by Φ to the (potential) KP equation

$$(4\, \phi_{t_3} - \phi_{xxx} - 6\, (\phi_x)^2)_x = 6\, [\phi_{t_2}, \phi_x] + 3\, \phi_{t_2 t_2}\ . \tag{11}$$

4 From $\mathcal{A}(P)$ to the AKNS hierarchy

Let $\mathcal{B}$ be an associative $\mathbb{K}$-algebra with unit I, and $\mathcal{B}_\lambda$ the algebra of formal series (in an indeterminate λ and its inverse)

$$X = \sum_{m \leq M} \lambda^m X_m \tag{12}$$

where $X_m \in \mathcal{B}$ and $M \in \mathbb{Z}$. We set

$$X_{\geq 0} := \sum_{0 \leq m \leq M} \lambda^m X_m\, , \qquad X_{<0} := X - X_{\geq 0} = \sum_{m<0} \lambda^m X_m\ . \tag{13}$$

Next we choose an element $V \in \mathcal{B}_\lambda$ of the form

$$V = v_0 + \lambda^{-1}\, v_1 + \lambda^{-2}\, v_2 + \lambda^{-3}\, v_3 + \dots\, , \qquad J := v_0 \tag{14}$$

with $J, v_m \in \mathcal{B}$. Note that $v_m = V_{-m}$, $m = 0, 1, \ldots$. A generalization of the well-known AKNS hierarchy (see also [4]) is then determined by

$$V_{t_n} = [(\lambda^n V^n)_{\geq 0}, V] = -[(\lambda^n V^n)_{<0}, V] =: \delta_n V \qquad n = 1, 2, \dots \tag{15}$$

which requires $J_{t_n} = 0$. By a standard argument, the flows commute. The first ($n = 1$) hierarchy equation is equivalent to

$$[J, v_{m+1}] + [v_1, v_m] = v_{m,x} \qquad m = 1, 2, \dots\ . \tag{16}$$

Next we define two maps $\ell, \mathbf{r} : \mathcal{A}(P) \to \mathcal{B}_\lambda$ via $\ell(P) = \mathbf{r}(P) = \lambda\, V$ and

$$\ell(\alpha \prec P) = -\ell(\alpha)_{<0}\, \lambda V\, , \qquad \ell(\alpha \bullet P) = \ell(\alpha)\, \lambda V \tag{17}$$

$$\mathbf{r}(P \prec \alpha) = -\lambda V\, \mathbf{r}(\alpha)_{\geq 0}\, , \qquad \mathbf{r}(P \bullet \alpha) = \lambda V\, \mathbf{r}(\alpha) \tag{18}$$

for all $\alpha \in \mathcal{A}(P)$. As a consequence, we have $\ell(P^{\bullet m}) = \mathbf{r}(P^{\bullet m}) = \lambda^m\, V^m$.[3] The map $\Phi : \mathcal{A}(P) \to \mathcal{B}$ defined by

$$\Phi(\alpha) := (\ell(\alpha)_{<0}\, \lambda V)_{\geq 0} = \ell(\alpha)_{-1}\, J \tag{19}$$

[3] Several properties of ℓ and $\mathbf{r}$ in the KP case have been derived in [2]. Most of them in fact remain valid if we replace L by $\lambda\, V$.

then satisfies $\Phi(\alpha) = (\mathbf{r}(\alpha)_{<0}\,\lambda V)_{\geq 0} = \mathbf{r}(\alpha)_{-1}\,J$ and

$$\Phi(P^{\bullet k}) = (\lambda^k V^k)_{-1}\,J = (V^k)_{-(k+1)}\,J\ . \tag{20}$$

In particular, we obtain $\Phi(P) = v_2\,J$ and

$$\Phi(P^{\bullet 2}) = (\{J, v_3\} + \{v_1, v_2\})\,J \tag{21}$$

$$\Phi(P^{\bullet 3}) = (\frac{1}{2}\{J, J, v_4\} + \{J, v_1, v_3\} + \frac{1}{2}\{J, v_2, v_2\} + \frac{1}{2}\{v_1, v_1, v_2\})\,J \tag{22}$$

where $\{a_1, \ldots, a_k\} := \sum_{\sigma \in \mathcal{S}_k} a_{\sigma(1)} \cdots a_{\sigma(k)}$ with the symmetric group $\mathcal{S}_k$ of order k. It can be shown that Φ has the following algebra homomorphism property,

$$\Phi(\alpha \hat{\times} \beta) = (\ell(\alpha)_{<0}\,\lambda V\,\mathbf{r}(\beta)_{<0})_{-1}\,J = \Phi(\alpha)\,\Phi(\beta) \tag{23}$$

for all $\alpha, \beta \in \mathcal{A}(P)$ (cf theorem 6.2 in [2]). Another important formula is[4]

$$\Phi(\alpha)_{t_n} = \delta_n \Phi(\alpha) = \Phi(P^{\bullet n} \circ \alpha) \tag{24}$$

where we imposed the flow equations $\delta_n = \partial_{t_n}$ on $\mathcal{B}_\lambda$. Applying Φ to the simple algebraic identities $P^{\bullet k} \circ P^{\bullet n} = P^{\bullet n} \circ P^{\bullet k}$ leads to the relations

$$(\Phi(P^{\bullet n}))_{t_k} = (\Phi(P^{\bullet k}))_{t_n} \tag{25}$$

(which in the KP case are trivially satisfied). The identity (10) is mapped by Φ to

$$\begin{aligned} &\Big(4\,v_{2,t_3} - v_{2,xxx} - 3\,(\{J, v_3\} + \{v_1, v_2\})_{t_2} - 6\,(v_2 J v_2)_x\Big)\,J \\ &+6\,[v_2 J, (\{J, v_3\} + \{v_1, v_2\})\,J] = 0\ . \end{aligned} \tag{26}$$

4.1 $V^2 = V$ reduction

For any polynomial $\mathcal{P}$ of V with coefficients in the center of $\mathcal{B}$, the constraint $\mathcal{P}(V) = 0$, which in particular requires $\mathcal{P}(J) = 0$, is preserved by the hierarchy (15). Let us consider the special case $V^2 = V$, which is equivalent to

$$v_m = \sum_{i=0}^{m} v_i\,v_{m-i} = \{J, v_m\} + \sum_{i=1}^{m-1} v_i\,v_{m-i} \qquad m = 0, 1, 2, \ldots\ . \tag{27}$$

We further assume that the first of the hierarchy equations (15), i.e. $V_x = [\lambda J + v_1, V]$ holds, and thus (16). Together with (27), this implies

$$v_{m+1} = -(v_{m,x} + \sum_{i=1}^{m} v_i\,v_{m+1-i} - [v_1, v_m])\,H \tag{28}$$

where $H := 2J - I$ which satisfies $H^2 = I$. This allows us to express v_m, $m > 1$, iteratively in terms of $u := v_1$ and its derivatives with respect to x,

$$v_2 = -(u_x + u^2)\,H\,, \qquad v_3 = u_{xx} - 2\,u^3 + [u, u_x] \tag{29}$$

$$v_4 = -(u_{xxx} + \{u, u_{xx}\} - 3\,\{u^2, u_x\} - (u_x)^2 - 3\,u^4)\,H \tag{30}$$

[4] This is the analog of the simplest case expressed by proposition 6.5 in [2].

etc. It is well-known that soliton equations emerge from the hierarchy (15) for $n > 1$. But now we show how to obtain them from identities in $\mathcal{I}$. Using the above results, the Φ-images (25) of algebraic identities for $n = 1$ and $k = 2, 3$ become

$$v_{2,y}\, J = v_{3,x}\, J\,, \qquad v_{2,t}\, J = v_{4,x}\, J \tag{31}$$

where $y := t_2$ and $t := t_3$. Let us choose

$$J = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad u = \begin{pmatrix} 0 & q \\ r & 0 \end{pmatrix} \tag{32}$$

where q and r are elements of a (not necessarily commutative) algebra. Then

$$v_2 = \begin{pmatrix} -qr & q_x \\ -r_x & rq \end{pmatrix}, \qquad v_3 = \begin{pmatrix} qr_x - q_x r & q_{xx} - 2\,qrq \\ r_{xx} - 2\,rqr & rq_x - r_x q \end{pmatrix} \tag{33}$$

$$v_4 = \begin{pmatrix} -qr_{xx} - q_{xx}r + q_x r_x + 3qrqr & -q_{xxx} + 3\,(q_x rq + qrq_x) \\ -r_{xxx} + 3\,(r_x qr + rqr_x) & rq_{xx} + r_{xx}q - r_x q_x - 3\,rqrq \end{pmatrix} \tag{34}$$

and equations (31) yield (after an integration)

$$(q_y - q_{xx} + 2\,qrq)\, r = 0\,, \qquad r_y + r_{xx} - 2\,rqr = 0 \tag{35}$$

$$(q_t - q_{xxx} + 3\,(q_x rq + qrq_x))\, r = 0\,, \qquad r_t - r_{xxx} + 3\,(r_x qr + rqr_x) = 0\,. \tag{36}$$

(35) is a system of coupled nonlinear Schrödinger equations. (36) yields with $r = 1$ the (noncommutative) KdV equation, and with $q = r$ the (noncommutative) mKdV equation. Moreover, (26) is satisfied as a consequence of (35) and (36).

4.2 $V^3 = I$ reduction

In this subsection we sketch another reduction: $V^3 = I$. Let

$$J = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \zeta & 0 \\ 0 & 0 & \zeta^2 \end{pmatrix}, \qquad u := v_1 = (1 + 2\,\zeta) \begin{pmatrix} 0 & r & q \\ q & 0 & r \\ r & q & 0 \end{pmatrix} \tag{37}$$

where ζ is a third root of unity (so that $\zeta^2 + \zeta + 1 = 0$) and $q, r \in \mathcal{B}$. (16) determines the non-diagonal part of v_m, the reduction the diagonal part. We obtain

$$v_2 = \begin{pmatrix} 3\,q\,r & \zeta\, r_x & \zeta/(1+\zeta)\, q_x \\ -\zeta\, q_x & -3\,(1+\zeta)\, q\, r & r_x \\ -\zeta/(1+\zeta)\, r_x & -q_x & -3/(1+\zeta)\, q\, r \end{pmatrix} \tag{38}$$

$$v_3 = \begin{pmatrix} (\zeta - 1)\, D/(1+\zeta) & -R/(2+\zeta) & \zeta Q/(1+2\zeta) \\ -Q/(2+\zeta) & -(1+2\zeta)\, D/(1+\zeta) & (1+\zeta) R/(\zeta - 1) \\ \zeta R/(1+2\zeta) & (1+\zeta) Q/(\zeta - 1) & (1-\zeta)\, D \end{pmatrix}. \tag{39}$$

where $Q := q_{xx} + 9q^2 r - 3rr_x$, $R := r_{xx} + 9qr^2 + 3qq_x$ and $D := q^3 + r^3 + qr_x - q_x r$. Next we compute (21) and evaluate the identity (25) for $n = 1$ and $k = 2$. Setting $t := (1 + 2\zeta)\, t_2$, this yields the following system of coupled Burgers equations

$$q_t - q_{xx} + 6\, r\, r_x = 0\,, \qquad r_t + r_{xx} + 6\, q\, q_x = 0\,. \tag{40}$$

5 From AKNS to KP

The existence of maps Φ_{KP} and Φ_{AKNS} which map identities in the algebra $\mathcal{A}(P)$ to equations of the KP, respectively AKNS hierarchy suggests a relation between the latter hierarchies. Let us see what happens if we identify their images. The equation $\Phi_{\mathrm{KP}}(P) = \Phi_{\mathrm{AKNS}}(P)$ reads

$$\phi_x = v_2 \, J \tag{41}$$

and, more generally, $\Phi_{\mathrm{KP}}(P^{\bullet n}) = \Phi_{\mathrm{AKNS}}(P^{\bullet n})$ means

$$\phi_{t_n} = (\lambda^n V^n)_{-1} \, J \, . \tag{42}$$

With the reduction treated in section 4.1, (41) becomes $\phi_x = -q\, r$, which indeed is a well-known (symmetry) constraint of the KP equation [6]. As a consequence of it, if q, r satisfy the AKNS equations (35) and (36), then ϕ satisfies the potential KP equation (11). (41) generalizes this relation to matrix (potential) KP equations. A thorough analysis of $\Phi_{\mathrm{KP}} = \Phi_{\mathrm{AKNS}}$ has still to be carried out, but we verified with the help of computer algebra in several examples that indeed matrix (potential) KP equations are satisfied by (41) as a consequence of the corresponding AKNS equations. We expect that this relation extends to the whole hierarchies. We plan to report on the relations between the abstract algebra $\mathcal{A}(P)$ and the KP and AKNS hierarchies sketched in this work in more detail in a separate publication.

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